6 Time evolution

6.1 Unitary evolution

The time evolution postulate of quantum mechanics states that, between measurements, the time evolution of the state vector of a closed quantum system Q is given by a linear differential equation called the **Schrödinger equation**: $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$. Here, $\hbar$ is the *reduced Planck constant*, we choose our units so that it's equal to one. $H \in \mathrm{Herm}(\mathcal{H}_\mathsf{Q})$ is the Hamiltonian - the observable corresponding to the total energy of a system. Solving the Schrödinger equation (for a time independent H) we obtain, for any $|\psi(t_1)\rangle$, $|\psi(t_2)\rangle = U(\Delta t)|\psi(t_1)\rangle$ where $U(\Delta t) := \exp(-iH\Delta t)$ is a *unitary* operator on $\mathcal{H}_\mathsf{Q}$, and $\Delta t := t_2 - t_1$. We won't talk about Hamiltonians again in this course. For us the important point is that the time evolution of a closed system is unitary and, furthermore, for *any* unitary transformation U we can find a Hamiltonian H and time interval Δt such that $U = \exp(-iH\Delta t)$. We will often imagine that we have sufficient control over systems to cause any unitary time evolution we like (by intervening to change the Hamiltonian). Sometimes this is not unrealistic: e.g. one can implement any unitary on a photon's polarisation (an example of a qubit) by passing the photon through various optical components.

Given a unitary time evolution U, we know how it acts on density operators corresponding to pure states of the system: $|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger$. Since the time evolved ensemble average of an ensemble of pure states should be the ensemble average of the time evolved pure state, density operators evolve according to $\rho \mapsto U\rho U^\dagger$.

6.2 Operations

It is not hard to come up with some realistic examples of non-unitary evolution.

1. Uncertainty about unitary evolution, e.g. with probability p_i unitary U_i occurred. $\rho \mapsto \sum_i p_i U_i \rho U_i^\dagger$ (relevant to modelling *noise* in quantum computers, for example).

2. A PVM (with more than one possible outcome) is performed on the system (see example sheet 1).

3. Adding a system: $\rho_\mathsf{Q} \mapsto \rho_\mathsf{Q} \otimes \sigma_\mathsf{R}$.

4. **Isometric evolution**: $\rho_\mathsf{A} \mapsto V_{\mathsf{B}\leftarrow\mathsf{A}}\rho_\mathsf{A}V^\dagger_{\mathsf{A}\leftarrow\mathsf{B}}$ where $V \in \mathcal{L}(\mathcal{H}_\mathsf{A}, \mathcal{H}_\mathsf{B})$ is an isometry, i.e. $V^\dagger_{\mathsf{A}\leftarrow\mathsf{B}}V_{\mathsf{B}\leftarrow\mathsf{A}} = \mathbb{1}_\mathsf{A}$. If $\sigma_\mathsf{R} = |\sigma\rangle\langle\sigma|_\mathsf{R}$ then adding a system is an example of isometric evolution with $V_{\mathsf{QR}\leftarrow\mathsf{R}} = \sum_{0\le i<d_\mathsf{Q}} |i\rangle_\mathsf{Q} \otimes |\sigma\rangle_\mathsf{R}\langle i|_\mathsf{Q} \in \mathcal{L}(\mathcal{H}_\mathsf{Q} \otimes \mathcal{H}_\mathsf{R}, \mathcal{H}_\mathsf{Q})$.

5. The **identity operation**, $\mathbf{id}^{\mathsf{B}\leftarrow\mathsf{A}} : X_\mathsf{A} \mapsto \mathbb{1}_{\mathsf{B}\leftarrow\mathsf{A}}X_\mathsf{A}\mathbb{1}^\dagger_{\mathsf{A}\leftarrow\mathsf{B}}$ is another simple example of isometric evolution.

6. Removing a system: $\rho_\mathsf{QR} \mapsto \mathrm{Tr}_\mathsf{R}\rho_\mathsf{QR}$.

7. Compositions of these, e.g. $\rho_\mathsf{QR} \mapsto \mathrm{Tr}_\mathsf{R}U_\mathsf{RQ}\rho_\mathsf{Q} \otimes \sigma_\mathsf{R}U^\dagger_\mathsf{RQ}$.

These are all examples of **operations**, that is, linear maps from one space of operators to another which are **completely positive** and **trace preserving**.

Definition 1. A linear map $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} : \mathcal{L}(\mathcal{H}_{\mathsf{A}}) \to \mathcal{L}(\mathcal{H}_{\mathsf{B}})$ is

1. **trace preserving (TP)** if for all $X_{\mathsf{A}} \in \mathcal{L}(\mathcal{H}_{\mathsf{A}})$ $\mathrm{Tr}\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}X_{\mathsf{A}} = \mathrm{Tr}X_{\mathsf{A}}$;
2. **positive** if for all $X_{\mathsf{A}} \in \mathcal{L}(\mathcal{H}_{\mathsf{A}})$ such that $X_{\mathsf{A}} \geq 0$, $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}X_{\mathsf{A}} \geq 0$;
3. **completely positive (CP)** if, for any system R, $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} \otimes \mathbf{id}^{\mathsf{R}\leftarrow\mathsf{R}}$ is positive.
4. An **operation** (or CPTP map) if it is completely positive and trace preserving.

♣♣ Give an example of a map which is positive but not completely positive.

Proposition 2. If $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ and $\mathcal{N}^{\mathsf{C}\leftarrow\mathsf{B}}$ are positive then their composition $\mathcal{N}^{\mathsf{C}\leftarrow\mathsf{B}}\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ is positive. If they are CP then their composition is CP. If they are TP then their composition is TP. Consequently, compositions of operations are operations.

Proposition 3. Given maps $\mathcal{M}_j^{\mathsf{B}\leftarrow\mathsf{A}}$, let $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} = \sum_i p_i \mathcal{M}_j^{\mathsf{B}\leftarrow\mathsf{A}}$ where $p_i \geq 0$ are real numbers. If the $\mathcal{M}_j^{\mathsf{B}\leftarrow\mathsf{A}}$ are positive then $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ is positive. If the $\mathcal{M}_j^{\mathsf{B}\leftarrow\mathsf{A}}$ are CP then $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ is CP.

Proposition 4. Maps of the form $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} : X_{\mathsf{A}} \mapsto ZX_{\mathsf{A}}Z^\dagger$, where $Z \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$, are CP.

Proof. If $X_{\mathsf{A}} \geq 0$, then $\langle\psi|_{\mathsf{B}} ZX_{\mathsf{A}}Z^\dagger|\psi\rangle_{\mathsf{B}} = \langle\psi'|_{\mathsf{A}}X_{\mathsf{A}}|\psi'\rangle_{\mathsf{A}} \geq 0$ where $|\psi'\rangle_{\mathsf{A}} = Z^\dagger|\psi\rangle_{\mathsf{B}}$ for all $|\psi\rangle_{\mathsf{B}}$, so the map is positive. Since $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} \otimes \mathbf{id}^{\mathsf{R}\leftarrow\mathsf{R}}X_{\mathsf{AR}} = (Z \otimes \mathbb{1}_{\mathsf{R}})X_{\mathsf{AR}}(Z \otimes \mathbb{1}_{\mathsf{R}})^\dagger$ is positive (by the same reasoning) for any R, $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ is completely positive. □

Proposition 5. The following classes of maps are operations:

1. Adding a system in a fixed state, uncorrelated to the existing system: $\rho_{\mathsf{Q}} \mapsto \rho_{\mathsf{Q}} \otimes \sigma_{\mathsf{R}}$.
2. Isometric evolution: $\rho_{\mathsf{A}} \mapsto V\rho_{\mathsf{A}}V^\dagger$, where $V \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$ and $V^\dagger V = \mathbb{1}_{\mathsf{A}}$.
3. Removing a system: $\rho_{\mathsf{QR}} \mapsto \mathrm{Tr}_{\mathsf{R}}\rho_{\mathsf{QR}}$.

Proof. That (2) is CP is a special case of Proposition 4. By decomposing σ as a convex combination of pure states we see that (1) can be written as a positive linear combination of isometries, so complete positivity follows from Props. 4 and 3. Looking at the third characterisation of the partial trace in the previous handout, we see that its complete positivity also follows from Props. 4 and 3. Adding a system is clearly TP; that isometric evolutions are TP follows from cyclicity of trace and $V^\dagger V = \mathbb{1}_{\mathsf{Q}}$. Partial trace is trace preserving simply because $\mathrm{Tr}_{\mathsf{Q}}\mathrm{Tr}_{\mathsf{R}} = \mathrm{Tr}_{\mathsf{QR}}$. □

Definition 6. For any Hilbert space $\mathcal{H}_{\mathsf{A}}$, we define a linear map $\mathrm{vec}_{\mathsf{A}} : \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathbb{C}) \to \mathcal{H}_{\mathsf{A}}$ by its action on the computational basis:

$$\text{For } a \in \{0, \ldots, d_{\mathsf{A}} - 1\}, \mathrm{vec}_{\mathsf{A}} : \langle a|_{\mathsf{A}} \mapsto |a\rangle_{\mathsf{A}}.$$

$\mathrm{vec}_{\mathsf{A}}$ is a bijection, whose inverse is $\mathrm{vec}_{\mathsf{A}}^{-1} : |a\rangle_{\mathsf{A}} \mapsto \langle a|_{\mathsf{A}}$.

Note that $\mathrm{vec}_{\mathsf{A}}\langle\psi|_{\mathsf{A}} = |\psi\rangle^*_{\mathsf{A}}$! If we apply $\mathrm{vec}_{\mathsf{A}}$ to $|i\rangle_{\mathsf{B}}\langle j|_{\mathsf{A}} \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}}) \cong \mathcal{H}_{\mathsf{B}} \otimes \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathbb{C})$, we get $\mathrm{vec}_{\mathsf{A}}|i\rangle_{\mathsf{B}}\langle j|_{\mathsf{A}} = |i\rangle_{\mathsf{B}} \otimes |j\rangle_{\mathsf{A}}$, and this extends by linearity to an isomorphism between $\mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$ and $\mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{A}}$.

Given any two systems A and A' of equal dimension d let $|\Phi^+\rangle_{\mathsf{AA}'} := \sum_{0\le j<d} |j\rangle_{\mathsf{A}} \otimes |j\rangle_{\mathsf{A}'}$ and $\Phi^+_{\mathsf{AA}'} := |\Phi^+\rangle\langle\Phi^+|_{\mathsf{AA}'}$, and let $|\phi^+\rangle_{\mathsf{AA}'} = |\Phi^+\rangle_{\mathsf{AA}'}/\sqrt{d}$.

Definition 7. Given a linear map $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} : \mathcal{L}(\mathcal{H}_{\mathsf{A}}) \to \mathcal{L}(\mathcal{H}_{\mathsf{B}})$, its **operator representation** in $\mathcal{L}(\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}})$ is defined to be

$$\mathbf{id}^{\mathsf{A}\leftarrow\mathsf{A}'}\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}\Phi^+_{\mathsf{AA}'} = \sum_{0\le j,k<d_{\mathsf{A}}} (\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}|j\rangle\langle k|_{\mathsf{A}})_{\mathsf{B}} \otimes |j\rangle\langle k|_{\mathsf{A}}$$

where A' is a system with the same dimension as A (the $\mathbf{id}^{\mathsf{A}\leftarrow\mathsf{A}'}$ is just for relabelling).

The action of $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ can be written in terms of its operator representation M_{BA}:

$$\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}X_{\mathsf{A}} = \mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}\left(\sum_{0\le j,k<d_{\mathsf{A}}} |j\rangle\langle j|_{\mathsf{A}}X_{\mathsf{A}}|k\rangle\langle k|_{\mathsf{A}}\right) \tag{6.1}$$

$$= \sum_{0\le j,k<d_{\mathsf{A}}} \left(\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}|j\rangle\langle k|_{\mathsf{A}}\right)_{\mathsf{B}} \mathrm{Tr}|j\rangle\langle k|_{\mathsf{A}}X^{\mathrm{T}}_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{A}}M_{\mathsf{BA}}\mathbb{1}_{\mathsf{B}} \otimes X^{\mathrm{T}}_{\mathsf{A}} \tag{6.2}$$

So, we have an isomorphism between the vector spaces $\mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathsf{A}}), \mathcal{L}(\mathcal{H}_{\mathsf{B}}))$ and $\mathcal{L}(\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}})$ (sometimes called "Channel-state duality" or the "Choi-Jamiołkowski isomorphism").

Proposition 8. The map $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} : X_{\mathsf{A}} \mapsto ZX_{\mathsf{A}}Z^\dagger$, where $Z \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$, has the operator representation $|\zeta\rangle\langle\zeta|_{\mathsf{BA}} \in \mathcal{L}(\mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{A}})$ where $|\zeta\rangle_{\mathsf{BA}} = \mathrm{vec}_{\mathsf{A}}Z$.

Proof. If $Z = \sum_{0\le b<d_{\mathsf{B}}, 0\le a<d_{\mathsf{A}}} z_{ba}|b\rangle_{\mathsf{B}}\langle a|_{\mathsf{A}}$, then $|\zeta\rangle_{\mathsf{BA}} = \mathrm{vec}_{\mathsf{A}}(Z) = \sum_{b,a} z_{ba}|b\rangle_{\mathsf{B}} \otimes |a\rangle_{\mathsf{A}}$, and $\mathrm{Tr}_{\mathsf{A}}|\zeta\rangle\langle\zeta|_{\mathsf{BA}}\mathbb{1}_{\mathsf{B}} \otimes X^{\mathrm{T}}_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{A}}\left[\sum_{b,a,b',a'} z_{ba}z^*_{b'a'}|b\rangle\langle b'|_{\mathsf{B}} \otimes |a\rangle\langle a'|_{\mathsf{A}}X^{\mathrm{T}}_{\mathsf{A}}\right]$
$= \sum_{b,a,b',a'} z_{ba}z^*_{b'a'}|b\rangle\langle b'|_{\mathsf{B}}\langle a|_{\mathsf{A}}X_{\mathsf{A}}|a'\rangle_{\mathsf{A}} = \left(\sum_{b,a} z_{ba}|b\rangle_{\mathsf{B}}\langle a|_{\mathsf{A}}\right) X_{\mathsf{A}} \left(\sum_{b',a'} z^*_{ba}|a'\rangle_{\mathsf{A}}\langle b'|_{\mathsf{B}}\right)$. □

Proposition 9 (Representations of CP maps.)**.** Let M_{BA} be the operator representation of a map $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$. The following statements are equivalent:

1. $M_{\mathsf{BA}} \ge 0$.
2. There is a set of maps $\{K_j \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}}) : j \in \{1, \ldots, n\}\}$ such that $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}X_{\mathsf{A}} = \sum_{j=1}^n K_jX_{\mathsf{A}}K_j^\dagger$.
3. There is a system E and map $Z \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{E}} \otimes \mathcal{H}_{\mathsf{B}})$ such that $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}X_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{E}}ZX_{\mathsf{A}}Z^\dagger$.
4. $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ is completely positive.

Proof. For (1) $\implies$ (2), we know that $M_{\mathsf{BA}} = \sum_j |\kappa_j\rangle\langle\kappa_j|_{\mathsf{BA}}$ for some $|\kappa_j\rangle_{\mathsf{BA}} \in \mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{A}}$ (from the eigendecomposition of M_{BA}, for instance). So, using Proposition 8,

$$\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}X_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{A}}M_{\mathsf{BA}}\mathbb{1}_{\mathsf{B}} \otimes X^{\mathrm{T}}_{\mathsf{A}} = \sum_j \mathrm{Tr}_{\mathsf{A}}|\kappa_j\rangle\langle\kappa_j|_{\mathsf{BA}}\mathbb{1}_{\mathsf{B}} \otimes X^{\mathrm{T}}_{\mathsf{A}} = \sum_j K_jX_{\mathsf{A}}K_j^\dagger$$

where $K_j = \mathrm{vec}_{\mathsf{A}}^{-1}|\kappa_j\rangle_{\mathsf{BA}}$. For (1) $\implies$ (3), we use that $M_{\mathsf{BA}} = \mathrm{Tr}_{\mathsf{E}}|\zeta\rangle\langle\zeta|_{\mathsf{EBA}}$ for some $|\zeta\rangle_{\mathsf{EBA}} \in \mathcal{H}_{\mathsf{E}} \otimes \mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{A}}$. Again using Proposition 8,

$$\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} X_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{A}} M_{\mathsf{BA}} \mathbb{1}_{\mathsf{B}} \otimes X_{\mathsf{A}}^{\mathrm{T}} = \mathrm{Tr}_{\mathsf{A}}(\mathrm{Tr}_{\mathsf{E}}|\zeta\rangle\langle\zeta|_{\mathsf{EBA}})\mathbb{1}_{\mathsf{B}} \otimes X_{\mathsf{A}}^{\mathrm{T}} \tag{6.3}$$

$$= \mathrm{Tr}_{\mathsf{E}}\mathrm{Tr}_{\mathsf{A}}|\zeta\rangle\langle\zeta|_{\mathsf{EBA}}\mathbb{1}_{\mathsf{EB}} \otimes X_{\mathsf{A}}^{\mathrm{T}} = \mathrm{Tr}_{\mathsf{E}} Z X_{\mathsf{A}} Z^\dagger \tag{6.4}$$

where $Z \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{E}} \otimes \mathcal{H}_{\mathsf{B}}) = \mathrm{vec}_{\mathsf{A}}^{-1}|\zeta\rangle_{\mathsf{EBA}}$.

(2) $\implies$ (4) by Propositions 3 and 4. (3) $\implies$ (4) follows from the fact that isometries and partial traces are CP and from the composition of CP maps being CP. That (4) $\implies$ (1) is immediate from the definitions of complete positivity and the operator representation. □

An expression of the form $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} X_{\mathsf{A}} = \sum_j K_j X_{\mathsf{A}} K_j^\dagger$ is known as a **Kraus decomposition**, and the K_j are called **Kraus operators**, for $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$. An expression of the form $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} X_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{E}} Z X_{\mathsf{A}} Z^\dagger$ is known as a **Stinespring representation** for $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$.

Proposition 10. Given $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} : X_{\mathsf{A}} \mapsto \sum_{j=1}^n K_j X_{\mathsf{A}} K_j^\dagger = \mathrm{Tr}_{\mathsf{E}} Z X_{\mathsf{A}} Z^\dagger$ where $K_j \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$ and $Z \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{E}})$ then

$$\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} \text{ is trace preserving} \iff \textstyle\sum_{j=1}^n K_j^\dagger K_j = \mathbb{1}_{\mathsf{A}} \iff Z \text{ is an isometry.}$$

♣♣ Prove this.

Remark 11. [Stinespring representation of an operation] Propositions 9 and 10 tell us that any operation (CPTP map) $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}}$ can be written $\mathcal{M}^{\mathsf{B}\leftarrow\mathsf{A}} X_{\mathsf{A}} = \mathrm{Tr}_{\mathsf{E}} V X_{\mathsf{A}} V^\dagger$ for some *isometry* $V \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{E}})$.

Proposition 12. Any isometric evolution with $V \in \mathcal{L}(\mathcal{H}_{\mathsf{A}}, \mathcal{H}_{\mathsf{B}})$ can be written

$$V X_{\mathsf{A}} V^\dagger = \mathrm{Tr}_{\mathsf{A}} U_{\mathsf{AB}} X_{\mathsf{A}} \otimes |0\rangle\langle 0|_{\mathsf{B}} U_{\mathsf{AB}}^\dagger \tag{6.5}$$

where U_{AB} is a unitary in $\mathcal{L}(\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}})$.

Proof. Since V is isometry, it can be written $V = \sum_{0\le j<d_{\mathsf{A}}} |v_j\rangle_{\mathsf{B}}\langle j|_{\mathsf{A}}$ where $\{|v_j\rangle_{\mathsf{B}} : 0 \le j < d_{\mathsf{A}}\}$ is an orthonormal set. Therefore, $\{|0\rangle_{\mathsf{A}} \otimes |v_j\rangle_{\mathsf{B}} : 0 \le j < d_{\mathsf{A}}\}$ is an orthonormal set in $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$. It is always possible to extend an orthonormal set to an orthonormal basis. Let $\mathfrak{B} = \{|0\rangle_{\mathsf{A}} \otimes |v_j\rangle_{\mathsf{B}} : 0 \le j < d_{\mathsf{A}}\} \cup \{|u_{kj}\rangle_{\mathsf{AB}} : 1 \le k < d_{\mathsf{B}}, 0 \le j < d_{\mathsf{A}}\}$ be such an extension, and let $U_{\mathsf{AB}} = \sum_{0\le j<d_{\mathsf{A}}} |0\rangle_{\mathsf{A}} \otimes |v_j\rangle_{\mathsf{B}}\langle j|_{\mathsf{A}} \otimes \langle 0|_{\mathsf{B}} + \sum_{0\le j<d_{\mathsf{A}}} \sum_{1\le k<d_{\mathsf{B}}} |u_{kj}\rangle_{\mathsf{AB}}\langle j|_{\mathsf{A}} \otimes \langle k|_{\mathsf{B}}$. This is a unitary operator in $\mathcal{L}(\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}})$ since it maps the computational basis for $\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ to the orthonormal basis $\mathfrak{B}$. Because $U_{\mathsf{AB}}|\psi\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} = |0\rangle_{\mathsf{A}} \otimes (V|\psi\rangle_{\mathsf{A}})_{\mathsf{B}}$,

$$\mathrm{Tr}_{\mathsf{A}} U_{\mathsf{AB}} |\psi\rangle\langle\psi|_{\mathsf{A}} \otimes |0\rangle\langle 0|_{\mathsf{B}} U_{\mathsf{AB}}^\dagger = V|\psi\rangle\langle\psi|_{\mathsf{A}} V^\dagger.$$

The result follows by linearity. □

From the last two results it follows that

Theorem 13. Any operation can be implemented by adding a system in a pure state, unitary evolution of the composite system, and removing a system.