10.4 Comments on the teleportation protocol

1. In the protocol, whatever the state ρ_{Q}, the result M of Alice's measurement on QA is uniformly distributed. Therefore M, by itself, doesn't tell us anything about the state of Q.

2. Without conditioning on the result of Alice's measurement, the state of B is the same immediately after the measurement as it was before. Namely, it is $\mathrm{Tr}_{\mathsf{A}}\phi^{+}_{\mathsf{AB}}$, the maximally mixed state. On learning M, Bob knows that the state of B is $Z_{\mathsf{B}}^{M_1}X_{\mathsf{B}}^{M_2}\rho_{\mathsf{B}}X_{\mathsf{B}}^{M_2}Z_{\mathsf{B}}^{M_1}$, which allows him to apply an appropriate unitary operation to recover ρ_{B}

11 Cloning and superluminal communication

11.1 Cloning allows superluminal communication

Suppose Alice has system A and Bob has system B and the state of AB in the state ϕ^{+}_{AB}. Suppose that if $M = 0$ Alice measures A in the computational basis, and if $M = 1$, Alice measures the PVM E_1 on A

$$E_1(0) = |+\rangle\langle+|_{\mathsf{A}}, E_1(1) = |-\rangle\langle-|_{\mathsf{A}}.$$

Let us call the result of Alice's measurement X. Suppose that Bob did have a cloning machine and that Alice and Bob use clocks to arrange that Bob's operations occur after Alice's measurement (note that in special relativity the only way this even makes sense is if the two events separated by a time-like interval).

At a time when he knows Alice has measured, Bob applies his cloning machine to B. If $M = 0$, then Bob's state is

$$\frac{1}{2}\left(|0\rangle\langle 0|^{\otimes 2} + |1\rangle\langle 1|^{\otimes 2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

while if $M = 1$, then Bob's state is

$$\frac{1}{2}\left(|+\rangle\langle +|^{\otimes 2} + |-\rangle\langle -|^{\otimes 2}\right) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

These states are not the same, so the success probability to distinguish them is better than one half. If Bob applies his cloning machine a greater number of times to obtain n copies of his state after Alice's measurement, then probability that he can successfully determine whether $M = 0$ or $M = 1$ from a measurement of the n-fold copy approaches one.

An argument of this kind was used in a proposal for a superluminal signalling system by Nick Herbert (Foundations of Physics Volume 12, Issue 12 , pp 1171-1179). There is a nice summary of this paper here `http://www.scientificamerican.com/article/mistakes-faster-than-light-telegraph-that-wasnt/`

11.2 Quantum mechanics doesn't allow superluminal communication

Suppose Alice has system A and Bob has system B and the state of AB is ρ_{AB}. Now suppose that Alice performs a measurement on A which yields a result X and leaves her with quantum system A'. We can represent this by an instrument $\mathcal{I}^{\mathsf{A}'\leftarrow\mathsf{A}}$. The probability distribution of the result is

$$P_X(x) = \mathrm{Tr}_{\mathsf{AB}}\mathcal{I}(x)^{\mathsf{A}'\leftarrow\mathsf{A}} \otimes \mathbf{id}^{\mathsf{B}\leftarrow\mathsf{B}}\rho_{\mathsf{AB}} = \mathrm{Tr}_{\mathsf{A}}\mathcal{I}(x)^{\mathsf{A}'\leftarrow\mathsf{A}}\mathrm{Tr}_{\mathsf{B}}\rho_{\mathsf{AB}},$$

so it only depends on the state of A, $\mathrm{Tr}_{\mathsf{B}}\rho_{\mathsf{AB}}$. Conditional on the result having a particular value, say $X = x$, the state of $\mathsf{A}'\mathsf{B}$ is $\mathcal{I}(x)^{\mathsf{A}'\leftarrow\mathsf{A}} \otimes \mathbf{id}^{\mathsf{B}\leftarrow\mathsf{B}}\rho_{\mathsf{AB}}/P_X(x)$ and the state of B, which we denote by $\rho(x)_{\mathsf{B}}$, is

$$\rho(x)_{\mathsf{B}} = \mathrm{Tr}_{\mathsf{A}'}\mathcal{I}(x)^{\mathsf{A}'\leftarrow\mathsf{A}} \otimes \mathbf{id}^{\mathsf{B}\leftarrow\mathsf{B}}\rho_{\mathsf{AB}}/P_X(x).$$

Suppose we have some linear map $\mathcal{N}^{\mathsf{A}'\leftarrow\mathsf{A}} : \mathcal{L}(\mathcal{H}_{\mathsf{A}}) \to \mathcal{L}(\mathcal{H}_{\mathsf{A}'})$ and operator $M_{\mathsf{AB}} \in \mathcal{L}(\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}})$. As always, we can write $M_{\mathsf{AB}} = \sum_x L_{\mathsf{A}}^{(x)} \otimes K_{\mathsf{B}}^{(x)}$, for some operators $L^{(x)}$ and $K^{(x)}$ (which are not necessarily positive). Let $E_{\mathsf{A}} = \mathcal{N}^\dagger \mathbb{1}_{\mathsf{A}'}$.

$$\mathrm{Tr}_{\mathsf{A}'}\mathcal{N}^{\mathsf{A}'\leftarrow\mathsf{A}} \otimes \mathbf{id}^{\mathsf{B}\leftarrow\mathsf{B}}M_{\mathsf{AB}} = \sum_x \mathrm{Tr}_{\mathsf{A}'}(\mathcal{N}^{\mathsf{A}'\leftarrow\mathsf{A}}L_{\mathsf{A}}^{(x)}) \otimes K_{\mathsf{B}}^{(x)} = \sum_x (\mathrm{Tr}\mathcal{N}^{\mathsf{A}'\leftarrow\mathsf{A}}L_{\mathsf{A}}^{(x)})K_{\mathsf{B}}^{(x)}$$
$$= \sum_x (\mathrm{Tr}E_{\mathsf{A}}L_{\mathsf{A}}^{(x)})K_{\mathsf{B}}^{(x)} = \mathrm{Tr}_{\mathsf{A}}\sum_x E_{\mathsf{A}}L_{\mathsf{A}}^{(x)} \otimes K_{\mathsf{B}}^{(x)} = \mathrm{Tr}_{\mathsf{A}}E_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}M_{\mathsf{AB}}.$$

Therefore,

$$\rho(x)_{\mathsf{B}} = \mathrm{Tr}_{\mathsf{A}}E(x)_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}\rho_{\mathsf{AB}}/P_X(x),$$

where $E(x)_{\mathsf{A}} = \mathcal{I}(x)^\dagger \mathbb{1}_{\mathsf{A}'}$, for all x. That is, E is the POVM for the instrument $\mathcal{I}$. Note that in this situation, where we want to know the state of a subsystem conditional on the outcome of a measurement on a *different* subsystem, the state only depends on the POVM for the measurement.

Without conditioning on the result of the measurement, then the state of B after the measurement is

$$\sum_x P_X(x)\rho(x)_{\mathsf{B}} = \sum_x \mathrm{Tr}_{\mathsf{A}}E(x)_{\mathsf{A}} \otimes \mathbb{1}_{\mathsf{B}}\rho_{\mathsf{AB}} = \mathrm{Tr}_{\mathsf{A}}\rho_{\mathsf{AB}},$$

because $\sum_x E(x)_{\mathsf{A}} = \mathbb{1}_{\mathsf{A}}$. This is the same as the state of B before the measurement. We conclude that if Bob knows nothing about the result of Alice's measurement, then he has no way of knowing which measurement was performed, or even if Alice performed any measurement at all. Since the predictions of quantum mechanics in the scenario described here are independent of Alice and Bob's location, this conclusion is reassuring - otherwise quantum mechanics would be predicting some form of superluminal communication!